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Algebraic evaluation of rational polynomials in one-loop amplitudes

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Abstract: One-loop amplitudes are to a large extent determined by their unitarity cuts in four dimensions. We show that the remaining rational terms can be obtained from the ultraviolet behaviour of the amplitude, and determine universal form factors for these rational parts by applying reduction techniques to the Feynman diagrammatic representation of the amplitude. The method is valid for massless and massive internal particles. We illustrate this method by evaluating the rational terms of the one-loop amplitudes for $gg \to H$, $\gamma\gamma \to \gamma\gamma$, $gg \to gg$,$\gamma\gamma \to ggg$ and $\gamma\gamma \to \gamma\gamma\gamma\gamma$.

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1. Introduction

The upcoming LHC experiment provides a strong phenomenological motivation [1] to develop formalisms which allow the successful evaluation of partonic multi-leg processes at one loop. Recently, many different methods have been proposed to deal with this highly complex task. Apart from algebraic reduction algorithms, which for multi-leg processes lead to a proliferation of terms and motivate numerical [2 – 7] or semi-numerical [8 – 16] treatments at some stage, twistor-space inspired methods [17 – 24] have stimulated a great deal of activity to complete the task – initiated already more than ten years ago [25 – 27] – to evaluate one-loop amplitudes elegantly from their unitarity cuts [28 – 30]. These algorithms are fully successful for supersymmetric amplitudes or special classes of helicity amplitudes, where the UV behaviour is tamed, but for general Standard Model amplitudes, it is difficult to obtain information on the so-called rational polynomials, which are induced by the ultraviolet behaviour of Feynman integrals. This was analysed in great detail by Bern, Dixon, Dunbar and Kosower in [26], where criteria for 4-dimensional cut constructibility of one-loop amplitudes were derived. We will refer to these criteria, called "uniqueness result" in [26], as the "BDDK-theorem" in the following. The application of unitarity cuts

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to calculate the cut-constructible part of non-supersymmetric amplitudes has seen a lot of progress recently, and lead already to some remarkable results, as for example the cut constructible part of the six gluon amplitude [24]. This amplitude, including the rational parts, also has been evaluated numerically at a certain phase space point [31].

Very recently, progress also has been made to determine the remaining rational ambiguities within the unitarity-based method, using the so-called "bootstrap approach" [37 – 40]. These ideas have lead to the successful determination of previously unknown multi-leg QCD amplitudes [41 – 43].

An alternative approach to the evaluation of the polynomial terms of helicity amplitudes in QCD is worked out in detail in [32 – 34], where the missing rational parts of the so far unknown six gluon helicity amplitudes have been given. The authors have used reduction formulae in Feynman parameter space, based on the work of [35, 36]. Their formalism is designed for gluon amplitudes and massless internal propagators.

In this article, we show that the rational polynomials can be obtained in a general way, for massive as well as massless amplitudes and arbitrary external particles, in isolation from the cut-constructible parts, using Feynman diagrammatic reduction techniques. Our method is based on the tensor reduction formalism presented in [12], where explicit representations for tensor form factors have been derived. Projecting on the ultraviolet sensitive part of these form factors, we obtain the rational parts. The projection leads to a considerable simplification of the reduction cascade, which yields relatively compact expressions. This defines a method which allows for the automated evaluation of the rational polynomials of arbitrary one-loop amplitudes.[1]

In this sense, the two approaches – construction of an amplitude by unitarity cuts and using Feynman diagrams together with tensor reduction – can be considered as complementary: for the polynomial part, the Feynman diagrammatic approach seems to be more straightforward, while for the remaining parts of the amplitude the unitarity-based method often leads to faster and more compact results. Such a combination of techniques already has been employed in [45] to obtain the one-loop amplitude for a Higgs boson plus four negative helicity gluons. Of course, for such a combined formalism to be universally applicable, the generalisation of the unitarity based methods to massive internal propagators is required.

The paper is organised as follows: First we give a definition of rational or non-cut-constructible terms of general one-loop amplitudes in section 2. As an illustration, we calculate the polynomial terms of some 3, 4, 5- and 6-point amplitudes in section 3. Section 4 contains our conclusions. In the appendix we provide formulae which are useful for the extraction of rational polynomials of IR divergent amplitudes.

2. Rational terms of one-loop amplitudes

In this section we propose a definition of the rational part of a one-loop amplitude. To disentangle rational polynomials of infrared and ultraviolet origin, it is convenient to define

[1] Work on a similar subject has also appeared very recently in [44].

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the rational part with respect to the corresponding IR regulated amplitude. As an IR regulator, we use off-shell momenta for the external legs of our tensor integrals or masses for internal fermion lines. In this way it is guaranteed that all poles in ϵ will be of ultraviolet nature and only terms which are related to ultraviolet divergences lead to rational polynomials in an amplitude. It turns out that in the resulting expressions, the limits to the original kinematics are well defined. Amplitudes which contain dimensionally regulated infrared divergences could be used directly as a starting point as well, however in this case the two- and three-point functions have to be treated in a different way, as will be discussed in the appendix.

2.1 Definition of rational parts of amplitudes

Each N-point amplitude Γ can be written as a linear combination of tensor Feynman integrals in $n = 4 - 2\epsilon$ dimensions. Schematically this can be denoted as

$$\Gamma = \sum C_{\mu_1 \dots \mu_R}(n, \{s_{ij}, m_k\})\; I_N^{n,\mu_1 \dots \mu_R}(\{s_{ij}, m_k\})\;. \tag{2.1}$$

For our purposes, Γ can be regarded as a complex valued function of the dimension n and the kinematical invariants s_{ij}, m_k. The tensor integrals are defined in momentum space as,[2]

$$I_N^{n,\mu_1 \dots \mu_R}(\{s_{ij}, m_k\}) = \int \frac{d^n k}{i\,\pi^{n/2}}\; \frac{k^{\mu_1}\; \dots k^{\mu_R}}{(q_1^2 - m_1^2 + i\delta) \dots (q_N^2 - m_N^2 + i\delta)}\;, \tag{2.2}$$

where $q_j = k + r_j$ are the propagator momenta and $r_j = p_1 + \dots + p_j$ are sums of external momenta. In [12] we have presented all the relevant formulae to perform a complete tensor reduction of a general N-point tensor integral of rank $R \leq N$. For $N \geq 6$ the reduction is purely algebraic, in the sense that these rank R N-point functions decay into a linear combination of $(N-1)$-point functions of rank R–1. For $N \leq 5$ all tensor form factors are evaluated in terms of an adequate basis. We do not repeat these formulae here, but give the essential definitions to keep the paper self-contained. Introducing Feynman parameters leads immediately to the following representation of the tensor integrals [46 – 48]

$$\begin{aligned} I_N^{n,\mu_1 \dots \mu_R}(\{s_{ij}, m_k\}) = (-1)^R \sum_{m=0}^{[R/2]} \left(-\frac{1}{2}\right)^m \sum_{j_1 \dots j_{r-2m}=1}^{N} \left[(g^{\cdot\cdot})^{\otimes m}\, r_{j_1 \cdot}^{\cdot} \cdots r_{j_R \cdot}^{\cdot}\right]^{\{\mu_1 \cdots \mu_R\}} \\ \times I_N^{n+2m}(j_1 \dots, j_{R-2m}\,; \{s_{ij}, m_k\})\;. \end{aligned} \tag{2.3}$$

The objects $I_N^{n+2m}(j_1 \dots, j_{R-2m}\,; \{s_{ij}, m_k\})$ are scalar integrals in $D = n + 2m$ ($m = 0, 1, 2, \dots$) dimensions with Feynman parameters in the numerator:

$$\begin{aligned} &I_N^D(j_1, \dots, j_R\,; \{s_{ij}, m_k\}) = \\ &\qquad (-1)^N \Gamma(N - \frac{D}{2}) \int \prod_{i=1}^{N} dz_i\, \delta(1 - \sum_{l=1}^{N} z_l)\; z_{j_1} \dots z_{j_R} \left(-\sum_{k,l=1}^{N} \mathcal{S}_{kl} z_k z_l / 2\right)^{D/2-N} \\ &\mathcal{S}_{kl} = (r_l - r_k)^2 - m_l^2 - m_k^2\;. \end{aligned} \tag{2.4}$$

[2] To make contact to eq.(2.1) of [12] we note that we have set $r_{a_1} = \dots = r_{a_N} = 0$ here for ease of notation.

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This leads to

$$\Gamma = \sum C(n, \{j_l\}, \{s_{ij}, m_k\}) I_N^{n+2m}(\{j_l\}; \{s_{ij}, m_k\}) \,, \tag{2.5}$$

where the sum runs over all different integrals making up the amplitude.

Note that the coefficients $C(n, \dots)$ depend on the way the numerators of the Feynman diagrams are evaluated, i.e. on the renormalisation scheme which defines the dimension of the Clifford algebra and the dimensionality of the internal and external particles (for a more detailed discussion see [49 – 51]). If chiral fermions are present or if one wants to define helicity amplitudes for partonic processes, the presence of γ_5 makes it necessary to distinguish 4-dimensional from $(n-4)$-dimensional contributions of the Dirac algebra and n-dimensional vectors. We note that due to the dimension splitting [52] of the loop momentum $k = \hat{k} + \tilde{k}$, where $\hat{k}$ is 4-dimensional and $\tilde{k}$ is $(n-4)$-dimensional, a few integrals with $\tilde{k}^2$-terms in the numerator have to be known, which can be mapped to higher dimensional integrals [27, 11]

$$\begin{aligned}
\int \frac{d^n k}{i\pi^{n/2}} \frac{(\tilde{k}\cdot\tilde{k})^\alpha}{(k^2-M^2)^N} &= (-1)^\alpha \frac{\Gamma(\alpha-\epsilon)}{\Gamma(1-\epsilon)} \frac{n-4}{2} \, I_N^{n+2\alpha} \\
\int \frac{d^n k}{i\pi^{n/2}} \frac{(\tilde{k}\cdot\tilde{k})^\alpha k^\mu k^\nu}{(k^2-M^2)^N} &= (-1)^{\alpha+1} \frac{\Gamma(\alpha-\epsilon)}{\Gamma(1-\epsilon)} g^{\mu\nu} \frac{n-4}{4} \frac{n+2\alpha}{n} \, I_N^{n+2+2\alpha} \,.
\end{aligned} \tag{2.6}$$

Therefore, these integrals contribute to the sum in eq. (2.5).

The rational part of an amplitude in general stems from two different sources: Firstly from a linear combination of $d = 4$ terms with the finite rational terms of Feynman parameter integrals, secondly from $(n-4)$-dimensional remnants of the Dirac algebra and the treatment of internal particles, which combine with UV pole parts of Feynman parameter integrals. Thus we define the *rational* part $\mathcal{R}$ of an amplitude Γ as[3]

$$\begin{aligned}
\mathcal{R}[\Gamma] = &\sum C(4, \{j_l\}, \{s_{ij}, m_k\}) \mathcal{R}[I_N^{n+2m}(\{j_l\}; \{s_{ij}, m_k\})] \\
&+(n-4) \sum C'(4, \{j_l\}, \{s_{ij}, m_k\}) \, \mathcal{P}[I_N^{n+2m}(\{j_l\}; \{s_{ij}, m_k\})]
\end{aligned} \tag{2.7}$$

with

$$C'(4, \{j_l\}, \{s_{ij}, m_k\}) = \frac{d}{dn} C(n, \{j_l\}, \{s_{ij}, m_k\})\Big|_{n=4} \,.$$

In eq. (2.7), $\mathcal{P}$ is the projector onto the pole part of the argument, i.e. the $1/\epsilon$-term in the ϵ-expansion. The action of $\mathcal{R}$ on Feynman parameter integrals is outlined in more detail below. It is the coefficient C' which governs the renormalisation scheme dependence.

In [12] we have given explicitly all the necessary formulae to reduce the scalar integrals with nontrivial numerators to n-dimensional one-, two- and three-point functions, $(n+2)$-dimensional three-and four-point functions and $(n+4)$-dimensional four-point functions.

[3] We assume that Γ is either a genuinely UV finite or an UV renormalised amplitude. Note that counterterms can be expressed in terms of one- and two-point functions.

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Schematically

$$I_N^{n+2m}(\{j_l\}) = \sum \beta_1\, I_1^n + \sum \beta_2(\{j_l\})\, I_2^n(1|j_1|j_1,j_2) + \sum \beta_{3a}(\{j_l\})\, I_3^n(1|j_1|j_1,j_2|j_1,j_2,j_3) + \sum \beta_{3b}(\{j_l\})\, I_3^{n+2}(1|j_1) + \sum \beta_{4a}(\{j_l\})\, I_4^{n+2}(1|j_1|j_1,j_2|j_1,j_2,j_3) + \sum \beta_{4b}(j)\, I_4^{n+4}(1|j)\ , \tag{2.8}$$

where the arguments $(1|j_1|j_1,j_2|j_1,j_2,j_3)$ denote integrals with up to three Feynman parameters in the numerator, and $I_N^n(1)$ are the genuine N-point scalar integrals, which will be denoted simply by I_N^n in the following. These integrals we call the "GOLEM integral representation".[4] It is defined by the property that inverse Gram determinants can be completely avoided by using such a representation. The coefficients $\beta_k(\{j_l\})$ are polynomial in the kinematical variables $\{s_{ij}, m_k^2\}$, they do not depend on the dimensionality n.

The GOLEM integral representation is a preferable starting point for a numerical evaluation of one-loop amplitudes. In algebraic approaches it is useful to reduce the GOLEM integrals further to a smaller integral basis which allows for an easy isolation of IR/UV divergencies. A convenient choice is to express each GOLEM integral by a linear combination of the scalar integrals I_1^n, I_2^n, I_3^n, I_4^{n+2}. All necessary formulae can be found in [12]. It is only in this further reduction step that an n-dependence enters into the coefficients:

$$I_N^{n+2m}(\{j_l\}) = \sum c_1(n)\, I_1^n + \sum c_2(n)\, I_2^n + \sum c_3(n)\, I_3^n + \sum c_4(n)\, I_4^{n+2}\ . \tag{2.9}$$

The summation is understood over the different kinematically allowed 1,2,3- and 4-point functions, which are defined by all possible propagator pinches of the corresponding N-point function on the left-hand side.

To completely define the operator $\mathcal{R}$ introduced above, we need to determine its action on a linear combination of 1,2,3 and 4-point functions of the form given above. Using again the rule

$$\mathcal{R}[c(n) I_N] = c(4)\, \mathcal{R}[I_N] + (n-4)\, c'(4) \mathcal{P}[I_N]\ , \tag{2.10}$$

one sees that the rational part of an arbitrary integral of type $I_N^{n+2m}(\{j_l\})$ is defined by the rational and the pole part of scalar integrals with trivial numerators. As UV poles can only occur for $D/2 + m \geq N$, all D=4 three-point and D=6 four-point functions are UV finite, i.e. their pole parts are zero:

$$\mathcal{P}[I_4^6] = 0 \quad , \quad \mathcal{P}[I_3^4] = 0 \quad . \tag{2.11}$$

Further, the BDDK [26] theorem tells us that all polynomial terms of such integrals, e.g. terms $\sim \pi^2/6$, are fully reconstructible by considering 4-dimensional cuts. This suggests to *define* the rational parts of these integrals to be zero:

$$\mathcal{R}[I_3^4] = 0 \quad , \quad \mathcal{R}[I_4^6] = 0 \quad . \tag{2.12}$$

[4] GOLEM stands for "General One-Loop Evaluator of Matrix elements" [16].

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A detailed description of how these functions are uniquely reconstructible from the asymptotic logarithmic behaviour of an expression can be found in [25, 26].

We define the pole- and rational parts of two-point functions for $s \neq 0$ by considering the following massive representation

$$I_2^n(s, m_1^2, m_2^2) = \frac{\Gamma(1+\epsilon)}{\epsilon} - \int_0^1 dx \log(-sx(1-x) + xm_1^2 + (1-x)m_2^2)$$

as

$$\mathcal{P}[I_2^n] = \frac{1}{\epsilon} \quad , \quad \mathcal{R}[I_2^n] = 0 \; . \tag{2.13}$$

Note that in the case $m_i = 0$, another natural definition could be $\mathcal{R}[I_2^n] = 2$, but these rational terms are directly related to the cut-constructible logarithmic terms.

In the case $s = 0$ the two-point function degenerates to combinations of one-point functions. The different cases are related to one-point functions in the following way:

$$\begin{aligned} I_2^n(0, m_1^2, m_2^2) &= \frac{1}{m_1^2 - m_2^2}\Big(I_1^n(m_1^2) - I_1^n(m_2^2)\Big) \\ I_2^n(0, 0, m^2) &= \frac{2}{n-2} I_2^n(0, m^2, m^2) \\ &= \frac{1}{m^2}\, I_1^n(m^2) \; . \end{aligned} \tag{2.14}$$

As up to $\mathcal{O}(\epsilon)$

$$I_1^n(m^2) = \frac{\Gamma(1+\epsilon)}{\epsilon\,(1-\epsilon)}\, m^2 - m^2 \, \log(m^2) \; ,$$

we define the the pole and rational part of the one-point function to match the non-logarithmic term, i.e.

$$\mathcal{P}[I_1^n(m^2)] = \frac{m^2}{\epsilon} \quad , \quad \mathcal{R}[I_1^n(m^2)] = m^2 \; . \tag{2.15}$$

Note that the rational parts of $I_2^n(0, 0, m^2)$ and $I_2^n(0, m^2, m^2)$ turn out to be different using this definition, on the other hand they still respect relation (2.14).

After having defined the pole and rational parts of the scalar integrals I_4^{n+2}, I_3^n, I_2^n and I_1^n, the rational part of an amplitude, Eq. (2.7), is fully determined. For a renormalised, IR-finite amplitude, we are now in the position to define the *cut-constructible* part of the amplitude Γ indirectly by

$$\mathcal{C}[\Gamma] = (1 - \mathcal{R})[\Gamma] \; . \tag{2.16}$$

In the general case one may use the definition

$$\mathcal{C}[\Gamma] = \lim_{\text{IR}} (1 - \mathcal{R})[\Gamma_{\text{IR regulated}}] \; . \tag{2.17}$$

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By an "IR regulated" version of the amplitude we mean here a representation where internal propagator masses or virtualities of external particles which are zero in the original amplitude are treated as non-zero, i.e. $m_j \neq 0$ or $s_j = p_j \cdot p_j \neq 0$, in tensor form factors. This renders the amplitude IR finite in the limit $n \to 4$ and the extraction of pole and rational parts is as in the IR finite case, if we demand $\lim_{\rm IR} \Gamma_{\rm IR\ regulated} = \Gamma$, where $\lim_{\rm IR}$ denotes the limits $m_j \to 0$ and/or $s_j \to 0$. In doing so it is of course crucial to avoid the expansion in $(n-4)$, which does not commute with the limits $m_j \to 0, s_j \to 0$. The IR-limits $m_j^2 \to 0$ or $s_j \to 0$ do *not* commute in general with the extraction of pole and rational terms. An example will be discussed below in subsection 3.3. In the appendix we provide explicit expressions for pole and rational parts of IR divergent tensor form factors, which can be used to define rational terms differently, without using this IR regularisation procedure.

We have checked that in the case of massless internal particles our definition of cut-constructibility leads to identical form factor expressions as the ones presented in [32] which were used to confirm known results for five- and six-gluon amplitudes derived with the methods of [26]. We stress that it is not necessary to have an off-shell representation of the full amplitude; the IR-regulated representations of the needed tensor form factors are sufficient to define the rational and pole parts we are looking for.

For later use, we also define the following operator:

$$\mathcal{U}[I] = (\mathcal{P} + \mathcal{R})[I] \ . \tag{2.18}$$

Note that the unitarity based methods [35, 36] are not yet generalised to the case of massive loop integrals. Our definition can easily be adapted to modified definitions of *cut-constructibility* for massive one-loop amplitudes by redefining rules (2.11)-(2.15), once such a formalism is developed.

For practical purposes, it is more convenient to produce purely rational expressions for tensor integrals directly once and for all, instead of reducing to scalar integrals first and then apply the operators $\mathcal{P}$ and $\mathcal{R}$, because this avoids an explosion of terms at intermediate stages of the calculation. Therefore, we give a list of the pole- and rational parts of the form factors for tensor integrals for $N = 2, 3$ and 4, which were needed in the applications below. As will be explained below, their knowledge is sufficient for the determination of the rational parts of any N-point amplitude, including amplitudes with massive internal particles.

2.2 Rational parts of 2-point form factors

Higher dimensional 2-point functions can be reduced to $(4-2\epsilon)$-dimensional ones by applying scalar integral reduction formulae [35, 36, 47, 12, 32].

$$\begin{aligned} I_2^{n+2}(s, m_1^2, m_2^2) &= \frac{1}{2s\,(n-1)}\Big[(-s + m_1^2 - m_2^2) I_1^n(m_2^2) + \\ &\quad + (-s - m_1^2 + m_2^2) I_1^n(m_1^2) + \lambda(s, m_1^2, m_2^2)\, I_2^n(s, m_1^2, m_2^2)\Big] \ , \\ \lambda(x, y, z) &= x^2 + y^2 + z^2 - 2(xy + yz + xz) \ . \end{aligned} \tag{2.19}$$

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Applying $\mathcal{U}$ to this formula and using rules (2.13),(2.15) yields, for the special case $m_1^2 = m_2^2 = m^2$,

$$\mathcal{U}[I_2^{n+2}(s, m^2, m^2)] = \frac{s}{6}\Big(\frac{1}{\epsilon} + \frac{2}{3}\Big) - m^2\left(\frac{1}{\epsilon} + 1\right) . \tag{2.20}$$

The rational parts of the 2-point form factors can be read off directly from the formulae in appendix A of [12]. The form factors are defined by

$$\begin{aligned} I_2^{n,\mu}(s, m_1^2, m_2^2) &= r^\mu\, A^{2,1} \\ I_2^{n,\mu_1\mu_2}(s, m_1^2, m_2^2) &= g^{\mu_1\,\mu_2}\, B^{2,2} + r^{\mu_1}\, r^{\mu_2}\, A^{2,2} \end{aligned}$$

For equal internal masses one obtains

$$\begin{aligned} \mathcal{U}[A^{2,1}(s, m^2, m^2)] &= -\frac{1}{2\epsilon} \\ \mathcal{U}[B^{2,2}(s, m^2, m^2)] &= -\frac{s}{12}\left(\frac{1}{\epsilon} + \frac{2}{3}\right) + \frac{m^2}{2}\left(\frac{1}{\epsilon} + 1\right) \\ \mathcal{U}[A^{2,2}(s, m^2, m^2)] &= \frac{1}{3}\left(\frac{1}{\epsilon} + \frac{1}{6}\right) . \end{aligned} \tag{2.21}$$

Formulas for different internal masses are obtained analogously, but are not listed here as they are rather lengthy and because they are not needed in the following.

2.3 Rational parts of 3-point form factors

Again, the rational parts can be obtained by applying $\mathcal{U} = \mathcal{R} + \mathcal{P}$ as defined above to the reduction formulae in section 5.1 of [12]. In order to give compact formulae for the rational part of the three point form factors, we will introduce the following matrix:

$$H_{l_1\,l_2} = \frac{b_{l_1}\, b_{l_2}}{B} - \mathcal{S}^{-1}_{l_1\,l_2} \ , \ l_1, l_2 \in \{1, 2, 3\} \ , \tag{2.22}$$

where $b_l = \sum_{k=1}^3 \mathcal{S}^{-1}_{lk}$ and $B = \sum_{k=1}^3 b_k$. Note that internal propagator masses are present through the matrix $\mathcal{S}$ defined in eq. (2.4). Applying momentum conservation $r_3 = p_1 + p_2 + p_3 = 0$ and defining $G_{lk} = 2\ r_l \cdot r_k$ for $l, k \in \{1, 2\}$, it is easy to see that the third minor of H is just G^{-1}:

$$G^{-1} = -\frac{1}{\lambda(s_1, s_2, s_3)}\begin{pmatrix} 2\, s_3 & s_2 - s_1 - s_3 \\ s_2 - s_1 - s_3 & 2\, s_1 \end{pmatrix} \tag{2.23}$$

with $s_j = p_j \cdot p_j$ and $\lambda(s_1, s_2, s_3)$ as defined in eq. (2.19). Then one can define the quantity

$$V_{l_1\,l_2\,l_3} = -\frac{1}{18}\left[H_{l_1\,l_2}\left(\frac{1}{1+\delta_{l_2\,l_3}} + \frac{1}{1+\delta_{l_1\,l_3}}\right) + 5\, H_{l_1\,l_2}\,\frac{b_{l_3}}{B}\ + 1 \leftrightarrow 3 + 2 \leftrightarrow 3\right]. \tag{2.24}$$

Here the ratios b_j/B, where

$$\frac{b_1}{B} = \frac{-s_3\,(s_1 + s_2 - s_3) + m_3^2\,(s_1 - s_2 - s_3) + 2\, s_3\, m_1^2 - m_2^2\,(s_1 - s_2 + s_3)}{\lambda(s_1, s_2, s_3)} \ ,$$

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and $b_2/B, b_3/B$ are obtained by cyclic permutations of indices, are well defined in the limit of massless internal propagators and light-like on-shell kinematics, as they behave like $1/\lambda(s_1, s_2, s_3)$, which is well defined as long as at least one s_j is non-zero. Thus H and V are both well defined for all relevant kinematical cases, i.e. $m_j^2 \to 0$ and/or $s_j \to 0$. Note however that these limits are *not* always equal to the finite parts of the corresponding integrals in the case where one or two of the s_j are vanishing, as on-shell limits and ϵ-expansion do not commute in general. This issue will be treated in detail in appendix A. The tensor form factors are defined by

$$\begin{aligned}
I_3^{n,\,\mu_1}(r_1, r_2, r_3 = 0, m_1, m_2, m_3) &= \sum_{j_1=1}^{2} A_{j_1}^{3,1}\, r_{j_1}^{\mu_1} \\
I_3^{n,\,\mu_1\mu_2}(r_1, r_2, r_3 = 0, m_1, m_2, m_3) &= B^{3,2}\, g^{\mu_1\mu_2} + \sum_{j_1,j_2=1}^{2} A_{j_1 j_2}^{3,2}\, r_{j_1}^{\mu_1} r_{j_2}^{\mu_2} \\
I_3^{n,\,\mu_1\mu_2\mu_3}(r_1, r_2, r_3 = 0, m_1, m_2, m_3) &= \sum_{j_1=1}^{2} B_{j_1}^{3,3}\, (g^{\mu_1\mu_2} r_{j_1}^{\mu_3} + 2 \text{ perms.}) + \\
&\quad + \sum_{j_1,j_2,j_3=1}^{2} A_{j_1 j_2 j_3}^{3,3}\, r_{j_1}^{\mu_1} r_{j_2}^{\mu_2} r_{j_3}^{\mu_3}\,, \qquad (2.25)
\end{aligned}$$

where we used momentum conservation to have $r_3 = 0$. Using the equations in section 5.1 of ref. [12], one gets the following pole- and rational parts of the form factors:

$$\begin{aligned}
\mathcal{U}[A_l^{3,1}] &= 0 \\
\mathcal{U}[B^{3,2}] &= \frac{1}{4}\left(\frac{1}{\epsilon} + 1\right) \qquad (2.26) \\
\mathcal{U}[A_{l_1 l_2}^{3,2}] &= -\frac{1}{2}\, H_{l_1 l_2} \\
\mathcal{U}[B_l^{3,3}] &= -\frac{1}{12}\left(\frac{1}{\epsilon} + \frac{2}{3} + \frac{b_l}{B}\right) \\
\mathcal{U}[A_{l_1 l_2 l_3}^{3,3}] &= -V_{l_1 l_2 l_3}\,. \qquad (2.27)
\end{aligned}$$

The rational part of the rank one tensor integral is identically zero. In the massless case these formulae are identical to the ones derived for massless internal particles in [32].

The rational parts of the $(n+2)$-dimensional three-point functions are implicitly defined, e.g. for the case of all masses equal one finds

$$\mathcal{U}[I_3^{n+2}(s_1, s_2, s_3, m^2, m^2, m^2)] = \mathcal{U}[-2\, B^{3,2}] = -\frac{1}{2}\left(\frac{1}{\epsilon} + 1\right)\,. \qquad (2.28)$$

2.4 Rational parts of 4-point form factors

From the preceding subsection and the way the form factors have been computed in ref. [12], it is clear that only the rank 3 and rank 4 form factors can have a non-zero rational part.

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The form factors are defined by [12]

$$I_4^{n,\,\mu_1\mu_2\mu_3}(r_1,r_2,r_3,r_4=0,m_1,m_2,m_3,m_4) = \sum_{j_1=1}^{3} B_{j_1}^{4,3}\,(g^{\mu_1\mu_2}r_{j_1}^{\mu_3}+2\text{ perms.}) + \sum_{j_1,j_2,j_3=1}^{3} A_{j_1j_2j_3}^{4,3}\,r_{j_1}^{\mu_1}r_{j_2}^{\mu_2}r_{j_3}^{\mu_3} \tag{2.29}$$

$$I_4^{n,\,\mu_1\mu_2\mu_3\mu_4}(r_1,r_2,r_3,r_4=0,m_1,m_2,m_3,m_4) = C^{4,4}(g^{\mu_1\mu_2}g^{\mu_3\mu_4}+2\text{ perms.}) + \sum_{j_1,j_2=1}^{3} B_{j_1j_2}^{4,4}\,(g^{\mu_1\mu_2}r_{j_1}^{\mu_3}r_{j_2}^{\mu_4}+5\text{ perms.}) + \sum_{j_1,j_2,j_3,j_4=1}^{3} A_{j_1j_2j_3j_4}^{4,4}\,r_{j_1}^{\mu_1}r_{j_2}^{\mu_2}r_{j_3}^{\mu_3}r_{j_4}^{\mu_4}\,. \tag{2.30}$$

For rank 3, we get

$$\begin{aligned}\mathcal{U}[B_l^{4,3}] &= 0\\ \mathcal{U}[A_{l_1\,l_2\,l_3}^{4,3}] &= -\frac{1}{6}\sum_{j\in S}\Big[H_{l_1\,j}\,H_{l_2\,l_3}^{\{j\}}\,\bar\delta_{j\,l_2}\,\bar\delta_{j\,l_3}+1\leftrightarrow 2+1\leftrightarrow 3\Big]\end{aligned} \tag{2.31}$$

and for rank 4:

$$\mathcal{U}[C^{4,4}] = \frac{1}{24\,\epsilon}+\frac{5}{72} \tag{2.32}$$

$$\mathcal{U}[B_{l_1l_2}^{4,4}] = -\frac{1}{12\,B}\sum_{j\in S} b_j\,H_{l_1\,l_2}^{\{j\}}\,\bar\delta_{j\,l_1}\,\bar\delta_{j\,l_2} \tag{2.33}$$

$$\begin{aligned}\mathcal{U}[A_{l_1l_2l_3l_4}^{4,4}] = {}& f^{4,4}(l_1,l_2;l_3,l_4)+f^{4,4}(l_1,l_3;l_2,l_4)+f^{4,4}(l_1,l_4;l_3,l_2)\\ &+f^{4,4}(l_2,l_3;l_1,l_4)+f^{4,4}(l_2,l_4;l_3,l_1)+f^{4,4}(l_3,l_4;l_1,l_2)\\ &+g^{4,4}(l_1;l_2,l_3,l_4)+g^{4,4}(l_2;l_1,l_3,l_4)\\ &+g^{4,4}(l_3;l_2,l_1,l_4)+g^{4,4}(l_4;l_2,l_3,l_1)\end{aligned} \tag{2.34}$$

$$f^{4,4}(l_1,l_2;l_3,l_4) = \frac{1}{12\,B}\sum_{j\in S}\bar\delta_{j\,l_3}\,\bar\delta_{j\,l_4}\left[H_{l_1\,l_2}\,b_j+\frac{1}{2}\,b_{l_1}\,H_{j\,l_2}+\frac{1}{2}\,b_{l_2}\,H_{l_1\,j}\right]\,H_{l_3\,l_4}^{\{j\}}$$

$$g^{4,4}(l_1;l_2,l_3,l_4) = -\frac{1}{4}\sum_{j\in S} H_{l_1\,j}\,V_{l_2\,l_3\,l_4}^{\{j\}}$$

$$\bar\delta_{jl} = 1-\delta_{jl} = \begin{cases}1 \text{ if } j\neq l\\ 0 \text{ if } j=l\end{cases}\,.$$

In these formulae S is the index set labelling the internal propagators of the four-point function. $H^{\{j\}}$ and $V^{\{j\}}$ are related to the 3-point kinematics obtained when omitting propagator j from this set [12]. For completeness we also list

$$\mathcal{U}[I_4^n]=0\,,\quad \mathcal{U}[I_4^{n+2}]=0\,,\quad \mathcal{U}[I_4^{n+4}]=\frac{1}{6\epsilon}+\frac{5}{18}\,. \tag{2.35}$$

2.5 Rational parts of 5-point form factors

With the results given above and the explicit representations of the 5-point form factors given in [12], it is manifest that up to rank 3, all 5-point functions have vanishing rational

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terms:

$$\mathcal{U}[I_5^n] = \mathcal{U}[I_5^{n,\mu_1}] = \mathcal{U}[I_5^{n,\mu_1\mu_2}] = \mathcal{U}[I_5^{n,\mu_1\mu_2\mu_3}] = 0\,. \tag{2.36}$$

The rational terms of the form factors for the rank 4 and rank 5 tensor integrals can directly be obtained from the explicit formulae given in section 6 of [12], which manifestly show that no integrals apart from the ones given in the previous subsections appear in the reduction. For $N \geq 6$, the reduction is purely algebraic anyway, i.e. involves only kinematic matrices until five-point functions are reached. Therefore, the formulae given in the previous subsections are sufficient to calculate the rational parts of *arbitrary* N-point amplitudes.

As was shown in [12], N-point functions up to rank N–2 are algebraically reducible to rank 3 five-point functions, therefore all the corresponding rational terms are zero. We have thus re-derived, within our formalism, a well known result of [26]: all N-point amplitudes which contain, in a convenient gauge, at most rank N–2 tensor integrals are cut-constructible.

3. Applications

In the following, five examples will be presented to illustrate our approach. The first is Higgs production by gluon fusion, the second scattering of light-by-light. Then we consider the 4-gluon amplitude which is IR and UV divergent. Finally we discuss a pentagon and a hexagon amplitude, $ggg\gamma\gamma \to 0$ and $\gamma\gamma\gamma\gamma\gamma\gamma \to 0$.

3.1 Example 1: Higgs production by gluon fusion

Higgs production by gluon fusion is mediated by massive quark loops. In the Standard Model, the top quark provides the leading contribution. Up to a trivial colour structure $\sim \delta_{ab}$ the amplitude is given by

$$\mathcal{M} = -\frac{m_t}{v}\frac{g_s^2}{(4\pi)^{n/2}}\int \frac{d^n k}{i\,\pi^{n/2}}\frac{\mathrm{tr}(\not\varepsilon_1(\not q_1 + m_t)(\not q_2 + m_t)\not\varepsilon_2(\not k + m_t))}{(q_1^2 - m_t^2)(q_2^2 - m_t^2)(k^2 - m_t^2)} \tag{3.1}$$

Here $q_j = k + r_j$ with $r_1 = -p_1$ and $r_2 = p_2$, where p_j are the light-like momenta of the gluons with polarisation vectors ε_j. Apart from m_t^2 the only non-vanishing Lorentz invariant variable is $2p_1 \cdot p_2 = s = m_H^2$. Working out the trace leads to

$$\begin{aligned}\mathcal{M} \sim\; & I_3^n(r_1, r_2, 0, m_t^2, m_t^2, m_t^2)\,(-\varepsilon_1\cdot\varepsilon_2 + 2\,\varepsilon_1\cdot p_2\,\varepsilon_2\cdot p_1 + 2\,m_t^2\,\varepsilon_1\cdot\varepsilon_2) \\ & + I_3^{n\,\mu\nu}(r_1, r_2, 0, m_t^2, m_t^2, m_t^2)\,(8\,\varepsilon_{1\,\mu}\varepsilon_{2\,\nu} - 2\,\varepsilon_1\cdot\varepsilon_2\,g_{\mu\nu})\end{aligned} \tag{3.2}$$

The scalar integral I_3^n does not contribute to the rational part of the amplitude, while the rank 2 tensor integral does. The pole and rational parts of the form factors for $I_3^{n\,\mu\nu}$ defined above turn out to be

$$\begin{aligned}\mathcal{U}[B^{3,2}] &= \frac{1+\epsilon}{4\,\epsilon} \\ \mathcal{U}[A_{11}^{3,2}] &= -\mathcal{U}[A_{12}^{3,2}] = \mathcal{U}[A_{22}^{3,2}] = -\frac{1}{2\,s}\,.\end{aligned} \tag{3.3}$$

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The decomposition of the Feynman diagram in terms of partly UV divergent tensor integrals made it necessary to work in dimensional regularisation, although the amplitude is finite. The rational parts of the amplitude are a result of products of UV $1/\epsilon$ poles and order ϵ remnants from the n-dimensional gamma algebra. Note that the rational part of the tensor functions is only a small part of the whole tensor integral. Applying the rules of section 2, the rational part is found to be

$$\mathcal{R}[\mathcal{M}] = \frac{\alpha_s}{\pi} \frac{m_t^2}{v} \frac{\mathrm{tr}(\mathcal{F}_1 \mathcal{F}_2)}{s} \,. \tag{3.4}$$

Here $\mathcal{F}_j^{\mu\nu} = p_j^\mu \varepsilon_j^\nu - p_j^\nu \varepsilon_j^\mu$ is the abelian part of the gluon field strength tensor.

3.2 Example 2: scattering of light-by-light

In QED, scattering of light-by-light is mediated by a closed electron loop. It is well known that the six box topologies making up the amplitude are related, such that it is sufficient to evaluate only one diagram with a given ordering of the external photons. The others can be obtained by all non-cyclic permutations of the photon momentum and polarisation vectors.

$$\mathcal{M} = \frac{e^4}{(4\pi)^{n/2}} \sum_{\sigma \in S_4/Z_4} \mathcal{G}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \tag{3.5}$$

In addition, due to charge invariance, one has $\mathcal{G}(1,2,3,4) = \mathcal{G}(1,4,3,2)$, $\mathcal{G}(1,3,4,2) = \mathcal{G}(1,2,4,3)$ and $\mathcal{G}(1,4,2,3) = \mathcal{G}(1,3,2,4)$. We evaluate now the rational polynomial of $\mathcal{M}$ using the polynomial tensor coefficients. The evaluation of the diagram

$$\mathcal{G}(1,2,3,4) = -\int \frac{d^n k}{i\,\pi^{n/2}} \frac{\mathrm{tr}(\not{\varepsilon}_1(\not{q}_1 + m_e)\not{\varepsilon}_2(\not{q}_2 + m_e)\not{\varepsilon}_3(\not{q}_3 + m_e)\not{\varepsilon}_4(\not{k} + m_e))}{(q_1^2 - m_e^2)(q_2^2 - m_e^2)(q_3^2 - m_e^2)(k^2 - m_e^2)} \tag{3.6}$$

involves four-point tensor integrals up to rank four. These are in general complicated functions, but the rational polynomials are actually very simple. Note that we are only interested in the massless case $m_e \to 0$. We only keep the electron mass as an infrared cutoff for the moment. Otherwise the massless on-shell 2-point functions, which are zero in dimensional regularisation, would spoil a clear separation of IR and UV problems. We give now the complete list of the polynomial part of the four-point tensor coefficients. For rank zero, one and two, no rational terms are present. The rank 3 and rank 4 tensor coefficients are defined in eq. (2.29) above. The rational polynomials of the tensor coefficients depend on the external kinematics. In our case, where $p_j \cdot p_j = 0$ for $j = 1,2,3,4$, we find in the limit $m_e \to 0$

$$\begin{aligned}
&\mathcal{U}[B_1^{4,3}] = \mathcal{U}[B_2^{4,3}] = \mathcal{U}[B_3^{4,3}] = 0 \\
&\mathcal{U}[A_{111}^{4,3}] = \mathcal{U}[A_{333}^{4,3}] = \frac{u-t}{2\,s\,t\,u} \\
&\mathcal{U}[A_{112}^{4,3}] = \mathcal{U}[A_{233}^{4,3}] = \frac{1}{2\,s\,u} \\
&\mathcal{U}[A_{113}^{4,3}] = \mathcal{U}[A_{122}^{4,3}] = \mathcal{U}[A_{133}^{4,3}] = \mathcal{U}[A_{223}^{4,3}] = -\mathcal{U}[A_{123}^{4,3}] = \frac{1}{2\,t\,u} \\
&\mathcal{U}[A_{222}^{4,3}] = \frac{u-s}{2\,s\,t\,u} \,,
\end{aligned} \tag{3.7}$$

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the remaining ones are defined by symmetry under exchange of the lower indices. The non cut-constructible parts of the rank four form factors are

$$
\begin{aligned}
&\mathcal{U}[C^{4,4}] = \frac{1}{24}\frac{1}{\epsilon} + \frac{5}{72} \\
&\mathcal{U}[B^{4,4}_{11}] = \mathcal{U}[B^{4,4}_{13}] = \mathcal{U}[B^{4,4}_{22}] = \mathcal{U}[B^{4,4}_{33}] = -\mathcal{U}[B^{4,4}_{12}] = -\mathcal{U}[B^{4,4}_{23}] = -\frac{1}{12\,u} \\
&\mathcal{U}[A^{4,4}_{1111}] = \mathcal{U}[A^{4,4}_{3333}] = \frac{1}{s\,t} - \frac{1}{s\,u} + \frac{1}{2\,u^2} \\
&\mathcal{U}[A^{4,4}_{1112}] = \mathcal{U}[A^{4,4}_{2333}] = \frac{1}{2\,s\,u} - \frac{1}{2\,u^2} \\
&\mathcal{U}[A^{4,4}_{1113}] = \mathcal{U}[A^{4,4}_{1333}] = -\frac{1}{2\,s\,t} - \frac{1}{2\,s\,u} + \frac{1}{2\,u^2} \\
&\mathcal{U}[A^{4,4}_{1122}] = \mathcal{U}[A^{4,4}_{2233}] = -\frac{1}{6\,s\,t} + \frac{1}{2\,u^2} \\
&\mathcal{U}[A^{4,4}_{1123}] = \mathcal{U}[A^{4,4}_{1233}] = \frac{1}{6\,s\,t} + \frac{1}{6\,s\,u} - \frac{1}{2\,u^2} \\
&\mathcal{U}[A^{4,4}_{1133}] = -\frac{1}{3\,s\,t} - \frac{1}{3\,s\,u} + \frac{1}{2\,u^2} \\
&\mathcal{U}[A^{4,4}_{1222}] = \mathcal{U}[A^{4,4}_{2223}] = -\frac{1}{2\,s\,t} - \frac{1}{2\,s\,u} - \frac{1}{2\,u^2} \\
&\mathcal{U}[A^{4,4}_{1223}] = \frac{1}{6\,s\,t} + \frac{1}{6\,s\,u} + \frac{1}{2\,u^2}\,,
\end{aligned}
\tag{3.8}
$$

the remaining ones are defined by symmetry. Evaluation of the rational part of eq. (3.6) is now straightforward. Using these form factors we find for the rational part of the sum of all graphs the manifestly gauge invariant result

$$
\begin{aligned}
&\mathcal{R}[\sum_{\sigma\in S_4/Z_4} \mathcal{G}(\sigma_1,\sigma_2,\sigma_3,\sigma_4)] = \\
&\quad= \frac{8}{3}\frac{\mathrm{tr}(\mathcal{F}_1\mathcal{F}_2)}{s}\frac{\mathrm{tr}(\mathcal{F}_3\mathcal{F}_4)}{s} + \frac{8}{3}\frac{\mathrm{tr}(\mathcal{F}_1\mathcal{F}_3)}{u}\frac{\mathrm{tr}(\mathcal{F}_2\mathcal{F}_4)}{u} + \frac{8}{3}\frac{\mathrm{tr}(\mathcal{F}_1\mathcal{F}_4)}{t}\frac{\mathrm{tr}(\mathcal{F}_2\mathcal{F}_3)}{t} \\
&\quad+\frac{64}{3}\frac{\mathrm{tr}(\mathcal{F}_1\mathcal{F}_2)}{s}\frac{p_4\cdot\mathcal{F}_3\cdot p_1\;\; p_3\cdot\mathcal{F}_4\cdot p_1}{s\,t\,u} - \frac{64}{3}\frac{\mathrm{tr}(\mathcal{F}_1\mathcal{F}_3)}{u}\frac{p_1\cdot\mathcal{F}_2\cdot p_3\;\; p_3\cdot\mathcal{F}_4\cdot p_1}{s\,t\,u} \\
&\quad+\frac{64}{3}\frac{\mathrm{tr}(\mathcal{F}_1\mathcal{F}_4)}{t}\frac{p_1\cdot\mathcal{F}_2\cdot p_3\;\; p_4\cdot\mathcal{F}_3\cdot p_1}{s\,t\,u} + \frac{64}{3}\frac{\mathrm{tr}(\mathcal{F}_2\mathcal{F}_3)}{t}\frac{p_2\cdot\mathcal{F}_1\cdot p_3\;\; p_3\cdot\mathcal{F}_4\cdot p_1}{s\,t\,u} \\
&\quad-\frac{64}{3}\frac{\mathrm{tr}(\mathcal{F}_2\mathcal{F}_4)}{u}\frac{p_2\cdot\mathcal{F}_1\cdot p_3\;\; p_4\cdot\mathcal{F}_3\cdot p_1}{s\,t\,u} + \frac{64}{3}\frac{\mathrm{tr}(\mathcal{F}_3\mathcal{F}_4)}{s}\frac{p_2\cdot\mathcal{F}_1\cdot p_3\;\; p_1\cdot\mathcal{F}_2\cdot p_3}{s\,t\,u} \\
&\quad+1024\,\frac{p_2\cdot\mathcal{F}_1\cdot p_3\;\; p_1\cdot\mathcal{F}_2\cdot p_3}{s\,t\,u}\frac{p_4\cdot\mathcal{F}_3\cdot p_1\;\; p_3\cdot\mathcal{F}_4\cdot p_1}{s\,t\,u}\,,
\end{aligned}
\tag{3.9}
$$

where $\mathcal{F}_j^{\mu\nu} = p_j^\mu\varepsilon_j^\nu - p_j^\nu\varepsilon_j^\mu$ is the electromagnetic field strength tensor. On the amplitude level the UV pole cancels and thus $\mathcal{R} = \mathcal{U}$. The result simplifies further if one specialises to helicity amplitudes using spinor helicity methods [53]. Due to parity invariance and Bose symmetry only the helicities $++++$, $+++-$ and $++--$ have to be considered.

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The rational polynomials of the three helicity amplitudes are then given by

$$\begin{aligned}
\mathcal{R}[\mathcal{M}^{++++}] &= \quad 8\alpha^2\, \varepsilon_1^+ \cdot \varepsilon_2^+\, \varepsilon_3^+ \cdot \varepsilon_4^+ \quad\quad\; = 8\,\alpha^2\, \tfrac{[12][34]}{\langle 12\rangle\langle 34\rangle} \\
\mathcal{R}[\mathcal{M}^{+++-}] &= 8\,\alpha^2\, \varepsilon_1^+ \cdot \varepsilon_2^+\, \varepsilon_3^+ \cdot p_1 \varepsilon_4^- \cdot p_1\, \tfrac{2s}{tu} = -8\,\alpha^2\, \tfrac{[12]\langle 14\rangle[13]}{\langle 12\rangle[14]\langle 13\rangle} \\
\mathcal{R}[\mathcal{M}^{++--}] &= -8\,\alpha^2\, \varepsilon_1^+ \cdot \varepsilon_2^+\, \varepsilon_3^- \cdot \varepsilon_4^- \quad\quad\; = -8\,\alpha^2\, \tfrac{[12]\langle 34\rangle}{\langle 12\rangle[34]}\,.
\end{aligned}$$

We note that up to a phase the rational parts of the different amplitudes are the same. For the first two helicity configurations, the rational terms are the full result. For completeness we also quote the full result of the $++--$ case [54, 55]:

$$\mathcal{M}^{++--} = -8\,\alpha^2\, \left(1 + \frac{t-u}{s}\log\left(\frac{t}{u}\right) + \frac{1}{2}\frac{u^2+t^2}{s^2}\Big[\log^2\left(\frac{t}{u}\right) + \pi^2\Big]\right)\, \frac{[12]\langle 34\rangle}{\langle 12\rangle[34]}\,. \tag{3.10}$$

3.3 Example 3: gluon-gluon scattering

To apply our formalism to an IR divergent amplitude we have chosen the 4-gluon amplitude as an example. In the following we consider the helicity amplitudes $++++$ and $++--$. Using spinor helicity methods one can replace the polarisation vectors by polynomial terms in the external momenta times a global phase [$\mathrm{tr}^{\pm}(\dots) = \mathrm{tr}(\dots)/2 \pm \mathrm{tr}(\gamma_5 \dots)/2$]:

$$\begin{aligned}
\varepsilon_1^{+\,\mu_1}\varepsilon_2^{+\,\mu_2}\varepsilon_3^{+\,\mu_3}\varepsilon_4^{+\,\mu_4} &= \frac{[21]}{\langle 12\rangle}\frac{[43]}{\langle 34\rangle}\frac{1}{2\,s^2}\mathrm{tr}^-(\not{k}_1\gamma^{\mu_2}\not{k}_2\gamma^{\mu_1})\,\mathrm{tr}^-(\not{k}_3\gamma^{\mu_4}\not{k}_4\gamma^{\mu_3}) \\
\varepsilon_1^{+\,\mu_1}\varepsilon_2^{+\,\mu_2}\varepsilon_3^{-\,\mu_3}\varepsilon_4^{-\,\mu_4} &= \frac{[21]}{\langle 12\rangle}\frac{\langle 34\rangle}{[43]}\frac{1}{2\,s^2}\mathrm{tr}^-(\not{k}_1\gamma^{\mu_2}\not{k}_2\gamma^{\mu_1})\,\mathrm{tr}^+(\not{k}_3\gamma^{\mu_4}\not{k}_4\gamma^{\mu_3})\,.
\end{aligned} \tag{3.11}$$

As we are using the 't Hooft-Veltman scheme for the computation, these expressions implicitly project onto the 4-dimensional part of n-dimensional objects in loop diagrams. At tree level one finds

$$\begin{aligned}
\mathcal{A}_{\mathrm{LO}}^{++++} &= \mathcal{A}_{\mathrm{LO}}^{+++-} = 0 \\
\mathcal{A}_{\mathrm{LO}}^{++--} &= -16\pi\,\alpha_s \frac{1}{N_C}\frac{[21]\langle 34\rangle}{\langle 12\rangle[43]}\hat{\mathcal{A}}_{\mathrm{LO}}^{++--} \\
\hat{\mathcal{A}}_{\mathrm{LO}}^{++--} &= +\mathcal{T}_{\mathrm{ad}}^{1234}\,\frac{s}{t} + \mathcal{T}_{\mathrm{ad}}^{1243}\,\frac{s}{u} + \mathcal{T}_{\mathrm{ad}}^{1324}\,\frac{s^2}{tu}\,,
\end{aligned} \tag{3.12}$$

with the colour objects defined through a trace of colour matrices in the adjoint representation $T^c_{ab} = if^{cab}$

$$\mathcal{T}_{\mathrm{ad}}^{1234} = \mathrm{tr}(T^{c_1}T^{c_2}T^{c_3}T^{c_4})\,. \tag{3.13}$$

We have computed the next-to-leading order contributions in two ways to outline the extraction of the rational terms for the IR divergent and IR regulated case. As the $++++$ case is IR and UV finite, both evaluations give the same result for this helicity configuration:

$$\begin{aligned}
\mathcal{A}_{\mathrm{NLO}}^{++++} &= -4\,\alpha_s^2 \frac{[21][43]}{\langle 12\rangle\langle 34\rangle}\hat{\mathcal{A}}_{\mathrm{NLO}}^{++++} \\
\hat{\mathcal{A}}_{\mathrm{NLO}}^{++++} &= \frac{1}{3}(\mathcal{T}_{\mathrm{ad}}^{1234} + \mathcal{T}_{\mathrm{ad}}^{1324} + \mathcal{T}_{\mathrm{ad}}^{1342})\,.
\end{aligned} \tag{3.14}$$

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The amplitude is defined by rational parts only. In the $++--$ case we again extract an overall factor:

$$\begin{aligned}
\mathcal{A}^{++--}_{\text{NLO}} &= -4\,\alpha_s^2\,\frac{[21]\langle 34\rangle}{\langle 12\rangle[43]}\hat{\mathcal{A}}^{++--}_{\text{NLO}} \\
\hat{\mathcal{A}}^{++--}_{\text{NLO}} &= +\frac{(s^2+t^2+u^2)^2}{2\,stu}\,\mathcal{T}^{1324}_{\text{ad}}\,I_4^6(u,t) + 2\frac{su}{t}\,\mathcal{T}^{1234}_{\text{ad}}\,I_4^6(s,t) + 2\frac{st}{u}\,\mathcal{T}^{1243}_{\text{ad}}\,I_4^6(s,u) \\
&\quad -2\left(\frac{s^2}{t}\,\mathcal{T}^{1234}_{\text{ad}} + \frac{s^2}{u}\,\mathcal{T}^{1243}_{\text{ad}}\right)\,I_3^n(s) \\
&\quad -2\left(s\,\mathcal{T}^{1234}_{\text{ad}} + \frac{s^2}{u}\,\mathcal{T}^{1324}_{\text{ad}}\right)\,I_3^n(t) - 2\left(s\,\mathcal{T}^{1243}_{\text{ad}} + \frac{s^2}{t}\,\mathcal{T}^{1324}_{\text{ad}}\right)\,I_3^n(u) \\
&\quad +\left(-\frac{11\,s}{3\,t}\,\mathcal{T}^{1234}_{\text{ad}} + \left(\frac{11\,s}{3\,t} + \frac{t-u}{s}\right)\mathcal{T}^{1324}_{\text{ad}}\right)\,I_2^n(t) \\
&\quad +\left(-\frac{11\,s}{3\,u}\,\mathcal{T}^{1243}_{\text{ad}} + \left(\frac{11\,s}{3\,u} + \frac{u-t}{s}\right)\mathcal{T}^{1324}_{\text{ad}}\right)\,I_2^n(u) \\
&\quad -\left(\frac{s}{9\,t}\,\mathcal{T}^{1234}_{\text{ad}} + \frac{s}{9\,u}\,\mathcal{T}^{1243}_{\text{ad}} + \left(\frac{s^2}{9\,tu}+1\right)\mathcal{T}^{1324}_{\text{ad}}\right)\,. \qquad (3.15)
\end{aligned}$$

In our definition we call the last line the rational part of the amplitude. Note that other polynomial contributions emerge after expanding the scalar integrals in ϵ. We have checked that our result is — up to trivial factors stemming from different conventions — identical to the amplitude representations provided in [57, 58].

If we compute the same amplitude using off-shell values for the external momenta, i.e. $k_1^2 = k_2^2 = k_3^2 = k_4^2 = m^2$, some scalar functions change to their off-shell counterparts, e.g. $I_3^n(s) \to I_3^n(s, m^2, m^2)$, $I_4^{n+2}(u,t) \to I_4^{n+2}(u,t,m^2,m^2,m^2,m^2)$, which are all IR finite now. The coefficients of the IR regulated basis integrals and the constant part are up to terms of order $\mathcal{O}(m^2)$ identical to the coefficients of the original basis integrals. The only difference stems from the no-scale 2-point functions which are replaced by their off-shell counterpart, $I_2^n(0) \to I_2^n(m^2)$. This leads to an additional term which is proportional to the leading order amplitude:

$$\left(\frac{32}{3} + \frac{4}{9}\epsilon + \mathcal{O}(m^2)\right)\,I_2^n(m^2)\,\hat{\mathcal{A}}^{++--}_{\text{LO}}\,. \qquad (3.16)$$

This additional term vanishes in the on-shell limit, as $I_2^n(0) = 0$ in dimensional regularisation. When taking the on-shell limit it has to be put to zero *before* expanding in ϵ, i.e. before the pole/rational part is extracted, otherwise the limit does not exist. This result defines the IR regulated version of the 4-gluon amplitude. The IR-limit $\lim_{\text{IR}}(\Gamma_{\text{IR regulated}}) = \Gamma$ is smooth as long as scalar integrals are not expanded in ϵ. For 2-point functions with $\lim_{\text{IR}} I_2^n = 0$ the rational/pole parts must not be isolated from the result before the limit is taken. In practical calculations it is very easy to take care of these terms separately. This reasoning shows that the rational part of an amplitude as defined above is not affected by IR divergences.

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3.4 The $\gamma\gamma ggg \to 0$ amplitude

Following [56] the helicity amplitudes can be written as

$$\mathcal{M}^{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5} = \frac{Q_q^2 g_s^3}{i\pi^2} f^{c_3c_4c_5} \mathcal{A}^{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5} \,. \tag{3.17}$$

Due to Furry's theorem only one colour structure $\sim f^{c_3c_4c_5}$ exists. The diagrammatic structure implies that this amplitude is not cut constructible, as rank four and five 5-point functions and rank three and four 4-point functions are present. Six independent helicity amplitudes exist. For three of them the cut-constructible part is identically zero, they are given by the rational part only:

$$\mathcal{R}[\mathcal{A}^{+++++}] = \mathcal{A}^{+++++} = -\frac{\mathrm{tr}(\mathcal{F}_1^+\mathcal{F}_2^+)\mathrm{tr}(\mathcal{F}_3^+\mathcal{F}_4^+\mathcal{F}_5^+)}{2\, s_{34}s_{45}s_{35}} \,. \tag{3.18}$$

Here $\mathcal{F}_j^{\mu\nu} = p_j^\mu \varepsilon_j^\nu - p_j^\nu \varepsilon_j^\mu$ is the abelian part of the gluon field strength tensor.

$$\mathcal{A}^{-++++} = \frac{\mathrm{tr}(\mathcal{F}_2^+\mathcal{F}_3^+)\mathrm{tr}(\mathcal{F}_4^+\mathcal{F}_5^+)}{s_{23}^2 s_{45}^2} \Big[C^{-++++}\; p_2 \cdot \mathcal{F}_1^- \cdot p_4 - (4 \leftrightarrow 5) \Big] \tag{3.19}$$

with the coefficient

$$C^{-++++} = -\frac{s_{15}s_{12}}{s_{24}s_{35}} - \frac{s_{15}}{s_{35}} + \frac{s_{23}}{s_{24}} - \frac{s_{15}}{s_{34}} \,. \tag{3.20}$$

Further

$$\mathcal{A}^{++++-} = \frac{\mathrm{tr}(\mathcal{F}_1^+\mathcal{F}_2^+)\mathrm{tr}(\mathcal{F}_3^+\mathcal{F}_4^+)}{s_{12}^2 s_{34}^2} \Big[C^{++++-}\; p_1 \cdot \mathcal{F}_5^- \cdot p_3 - (3 \leftrightarrow 4) \Big] \tag{3.21}$$

with the coefficient

$$\begin{aligned} C^{++++-} = &-\frac{s_{45}s_{13}s_{14}}{s_{35}s_{15}s_{24}} - \frac{s_{13}s_{45}}{s_{15}s_{35}} + \frac{s_{45}^2}{s_{15}s_{24}} - \frac{s_{12}^2 + s_{45}^2 - s_{12}s_{45}}{s_{35}s_{15}} + \frac{s_{13}s_{15}}{s_{23}s_{45}} + \frac{s_{13} - s_{34}}{s_{23}} \\ &- \frac{s_{34}s_{45}}{s_{23}s_{15}} + \frac{s_{15} - s_{25}}{s_{45}} - \frac{s_{23} + s_{35}}{s_{13}} - \frac{s_{23}s_{25}}{s_{13}s_{45}} + \frac{s_{34} + s_{12}}{s_{15}} \,. \end{aligned} \tag{3.22}$$

The results for the helicity amplitudes $\mathcal{A}^{--+++}$, $\mathcal{A}^{+++--}$ and $\mathcal{A}^{-+++-}$ contain also contributions from the cuts. The full result for these amplitudes can be found in [56]. We have verified that we get the same result for the rational terms, called $\mathcal{A}_1^{--+++}$, $\mathcal{A}_1^{+++--}$ and $\mathcal{A}_1^{-+++-}$ there, using our algebraic implementation of the rational polynomials defined by our formalism.

3.5 The 6-photon amplitude

Due to Bose symmetry and parity invariance, only four independent helicity amplitudes have to be evaluated, out of which two, the "all plus" and the "one minus" amplitudes identically vanish [59]. We evaluated the 6-photon amplitudes along the lines of the 4-photon case, with the difference that after taking the trace, all reducible scalar products in the numerator were cancelled directly. We have verified that the rational parts of

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$\mathcal{M}^{++++++}$ and $\mathcal{M}^{+++++-}$ evaluate to zero using our formalism. For the non-vanishing amplitudes, we find

$$\mathcal{R}[\mathcal{M}^{++++--}] = 0 \tag{3.23}$$

$$\mathcal{R}[\mathcal{M}^{+++---}] = 0\,, \tag{3.24}$$

where eq. (3.23) is already known from the analytic result [60]. The results have been obtained by two independent calculations, one based on the approach outlined in section 2, the other one based on IR divergent form factors (see appendix A). We observe that for the evaluation of the rational parts, form factors for at most rank 4 four-point functions were needed. Kinematically they are of the same complexity as the ones used for the recent evaluation of the rational parts of the six gluon amplitude [34].

4. Conclusions

We have presented a formalism to evaluate the rational polynomials of arbitrary one-loop N-point amplitudes. It is based on a tensor form factor representation derived in [12]. The definition of rational parts of these form factors induces a definition of the rational part of the amplitude.

To disentangle contributions form UV and IR poles, we define first the rational polynomials with respect to IR-regulated amplitudes. We obtain compact expressions for the rational and pole terms of the tensor form factors which allow for an on-shell limit afterwards. In this way it is clear that the polynomial part of an amplitude origins only from the UV behaviour of the amplitude. This procedure defines rational polynomials of general one-loop amplitudes corresponding to the definition advocated in the literature [26, 32]. In addition, we give all the formulae to work with form factors which contain IR divergences in the appendix. For IR finite amplitudes both approaches obviously lead to the same result.

Both approaches were implemented in algebraic manipulation programs to allow for the fully automated evaluation of the rational parts of one-loop amplitudes starting from Feynman diagrams. The formalism has been applied to the evaluation of the amplitudes for $gg \to H$, $\gamma\gamma \to \gamma\gamma$, $gg \to gg$, $\gamma\gamma \to ggg$ and $\gamma\gamma \to \gamma\gamma\gamma\gamma$. For the first four examples we recover the well-known results. For the six-photon amplitude we have proven by explicit analytical computation that the rational terms of all Feynman diagrams add up to zero for all helicity configurations.

Our implementation of this formalism is designed for arbitrary processes with up to six external legs, including massive particles, and makes it possible to obtain rational terms for phenomenologically relevant partonic amplitudes in an automated way. Note that numerical instabilities are typically mild in terms which are free of logarithms and dilogarithms. The method is thus a complement to the unitarity based techniques, which lead in general to compact representations for coefficients of non-polynomial terms. Combining both methods thus might be a very fruitful starting point for a highly effective method to evaluate complex multi-leg one-loop amplitudes.

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	$\mathcal{U}[I]$	$\mathcal{V}$	W
$I_3^n(0,0,X)$	$\mathcal{U}[I_3^n]$	0	0
$I_3^n(1;0,0,X)$	$-\frac{1}{X}\mathcal{U}[I_2^n]$	0	0
$I_3^n(2;0,0,X)$	$\mathcal{U}[I_3^n]+\frac{2}{X}\mathcal{U}[I_2^n]$	0	0
$I_3^n(3;0,0,X)$	$-\frac{1}{X}\mathcal{U}[I_2^n]$	0	0
$I_3^n(1,1;0,0,X)$	$-\frac{1}{2X}\mathcal{U}[I_2^n]$	0	0
$I_3^n(2,2;0,0,X)$	$\mathcal{U}[I_3^n]+\frac{3}{X}\mathcal{U}[I_2^n]+\frac{1}{X}$	$\frac{1}{X}$	0
$I_3^n(3,3;0,0,X)$	$-\frac{1}{2X}\mathcal{U}[I_2^n]$	0	0
$I_3^n(1,2;0,0,X)$	$-\frac{1}{2X}\mathcal{U}[I_2^n]-\frac{1}{2X}$	$-\frac{1}{2X}$	0
$I_3^n(1,3;0,0,X)$	$\frac{1}{2X}$	$\frac{1}{2X}$	0
$I_3^n(2,3;0,0,X)$	$-\frac{1}{2X}\mathcal{U}[I_2^n]-\frac{1}{2X}$	$-\frac{1}{2X}$	0
$I_3^n(1,1,1;0,0,X)$	$-\frac{1}{3X}\mathcal{U}[I_2^n]-\frac{1}{18X}$	0	$-\frac{1}{18X}$
$I_3^n(2,2,2;0,0,X)$	$\mathcal{U}[I_3^n]+\frac{11}{3X}\mathcal{U}[I_2^n]+\frac{19}{9X}$	$\frac{2}{X}$	$\frac{1}{9X}$
$I_3^n(3,3,3;0,0,X)$	$-\frac{1}{3X}\mathcal{U}[I_2^n]-\frac{1}{18X}$	0	$-\frac{1}{18X}$
$I_3^n(1,1,2;0,0,X)$	$-\frac{1}{6X}\mathcal{U}[I_2^n]-\frac{1}{9X}$	$-\frac{1}{6X}$	$\frac{1}{18X}$
$I_3^n(1,2,2;0,0,X)$	$-\frac{1}{3X}\mathcal{U}[I_2^n]-\frac{5}{9X}$	$-\frac{1}{2X}$	$-\frac{1}{18X}$
$I_3^n(1,1,3;0,0,X)$	$\frac{1}{6X}$	$\frac{1}{6X}$	0
$I_3^n(2,2,3;0,0,X)$	$-\frac{1}{3X}\mathcal{U}[I_2^n]-\frac{5}{9X}$	$-\frac{1}{2X}$	$-\frac{1}{18X}$
$I_3^n(1,3,3;0,0,X)$	$\frac{1}{6X}$	$\frac{1}{6X}$	0
$I_3^n(2,3,3;0,0,X)$	$-\frac{1}{6X}\mathcal{U}[I_2^n]-\frac{1}{9X}$	$-\frac{1}{6X}$	$\frac{1}{18X}$
$I_3^n(1,2,3;0,0,X)$	$\frac{1}{6X}$	$\frac{1}{6X}$	0

Table 1: Rational and pole parts of three-point functions with two light-like legs and up to three Feynman parameters in the numerator. $\mathcal{U}$ is the operator extracting the pole and rational parts of the integral. The $1/\epsilon$ and $1/\epsilon^2$ poles have been absorbed in the terms proportional to $\mathcal{U}[I_2^n]$ and $\mathcal{U}[I_3^n]$, respectively. $\mathcal{V}$ denotes the value obtained from the corresponding expression for the off-shell integral, in the limit where two legs go on-shell. W is the difference $\mathcal{U}[I]-\mathcal{V}$, where the pole terms have been set to zero.

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A. Rational parts of IR divergent integrals

As an alternative to the approach outlined above, where IR regulated amplitudes were considered, one can also define form factors for the rational parts in the presence of infrared poles. In section 2, we first apply the operator which extracts the rational parts and then take the on-shell limits of the form factors F:

$$\lim_{\mathrm{IR}} (\mathcal{P} + \mathcal{R}) \left[F(\text{IR-regulated}) \right] . \tag{A.1}$$

Another possibility is to apply the operators $\mathcal{P}$ and $\mathcal{R}$ directly to the potentially IR-divergent form factors, i.e. do the operation

$$(\mathcal{P} + \mathcal{R}) \left[F(\text{IR-divergent}) \right] . \tag{A.2}$$

In an infrared finite amplitude, of course all finite terms coming from the expansion of ϵ-dependent terms combined with $1/\epsilon_{\mathrm{IR}}$ poles finally have to cancel, such that the *remaining* finite polynomial parts are identical to the ones obtained by procedure (A.1), after summing over all contributions. In an infrared divergent amplitude, the finite remainders are related to the choice of the factorisation scheme.

In order to be able to define the polynomial part of divergent amplitudes, it is necessary to single out the contributions which come from the expansion of the poles. The results of operation (A.2) for the divergent three-point functions with Feynman parameters in the numerator, shown in Tables 1 to 3 below, are given in a form which allows to isolate these contributions immediately.

The definition of the rational parts of expressions containing $1/\epsilon^2$ and $1/\epsilon$ poles is of course linked to the overall ϵ-dependent factors which have been extracted. We single out the pole contributions in terms of $\mathcal{U}[I_3^n(0,0,X)]$ and $\mathcal{U}[I_2^n]$, i.e. all double poles have been absorbed into the scalar three-point function with two light-like legs, depending only on the invariant X, and all single poles have been absorbed into I_2^n, where

$$I_2^n \equiv I_2^n(X) = \frac{\tilde{r}_\Gamma}{\epsilon}(-X)^{-\epsilon} \tag{A.3}$$

$$I_3^n \equiv I_3^n(0,0,X) = \frac{\tilde{r}_\Gamma}{\epsilon^2}\frac{(1-2\epsilon)}{X}(-X)^{-\epsilon} \tag{A.4}$$

$$\tilde{r}_\Gamma = \frac{\Gamma(1+\epsilon)\,\Gamma^2(1-\epsilon)}{\Gamma(2-2\,\epsilon)} = \frac{r_\Gamma}{1-2\epsilon} . \tag{A.5}$$

Extracting an overall factor $\tilde{r}_\Gamma$ instead of r_Γ from *all* integrals, we have $\mathcal{R}[I_2^n] = 0$, which is more convenient for our purposes than extracting an overall factor $\tilde{r}_\Gamma$, which would imply $\mathcal{R}[I_2^n] = 2$.

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	$\mathcal{U}[I]$	$\mathcal{V}$	W
$I_3^n(0,X,Y)$	0	0	0
$I_3^n(1;0,X,Y)$	$\frac{1}{X-Y}\,\mathcal{U}[I_2^n]$	0	0
$I_3^n(2;0,X,Y)$	$\frac{1}{Y-X}\,\mathcal{U}[I_2^n]$	0	0
$I_3^n(3;0,X,Y)$	0	0	0
$I_3^n(1,1;0,X,Y)$	$\frac{3X-Y}{2(X-Y)^2}\,\mathcal{U}[I_2^n]+\frac{X}{(X-Y)^2}$	$\frac{X}{(X-Y)^2}$	0
$I_3^n(2,2;0,X,Y)$	$\frac{3Y-X}{2(X-Y)^2}\,\mathcal{U}[I_2^n]+\frac{Y}{(X-Y)^2}$	$\frac{Y}{(X-Y)^2}$	0
$I_3^n(3,3;0,X,Y)$	0	0	0
$I_3^n(1,2;0,X,Y)$	$-\frac{X+Y}{2(X-Y)^2}\;(\mathcal{U}[I_2^n]+1)$	$-\frac{X+Y}{2(X-Y)^2}$	0
$I_3^n(1,3;0,X,Y)$	$-\frac{1}{2(X-Y)}$	$-\frac{1}{2(X-Y)}$	0
$I_3^n(2,3;0,X,Y)$	$-\frac{1}{2(Y-X)}$	$-\frac{1}{2(Y-X)}$	0
$I_3^n(1,1,1;0,X,Y)$	$\frac{11X^2-7XY+2Y^2}{6(X-Y)^3}\,\mathcal{U}[I_2^n]+\frac{37X^2-8XY+Y^2}{18(X-Y)^3}$	$\frac{X(6X-Y)}{3(X-Y)^3}$	$\frac{1}{18(X-Y)}$
$I_3^n(2,2,2;0,X,Y)$	$\frac{11Y^2-7XY+2X^2}{6(Y-X)^3}\,\mathcal{U}[I_2^n]+\frac{37Y^2-8XY+X^2}{18(Y-X)^3}$	$\frac{Y(6Y-X)}{3(Y-X)^3}$	$\frac{1}{18(Y-X)}$
$I_3^n(3,3,3;0,X,Y)$	0	0	0
$I_3^n(1,1,2;0,X,Y)$	$\frac{-2X^2-5XY+Y^2}{6(X-Y)^3}\,\mathcal{U}[I_2^n]+\frac{-5X^2-11XY+Y^2}{9(X-Y)^3}$	$\frac{-3X^2-8XY+Y^2}{6(X-Y)^3}$	$-\frac{1}{18(X-Y)}$
$I_3^n(1,2,2;0,X,Y)$	$\frac{-2Y^2-5XY+X^2}{6(Y-X)^3}\,\mathcal{U}[I_2^n]+\frac{-5Y^2-11XY+X^2}{9(Y-X)^3}$	$\frac{-3Y^2-8XY+X^2}{6(Y-X)^3}$	$-\frac{1}{18(Y-X)}$
$I_3^n(1,1,3;0,X,Y)$	$\frac{-3X+Y}{6(X-Y)^2}$	$\frac{-3X+Y}{6(X-Y)^2}$	0
$I_3^n(2,2,3;0,X,Y)$	$\frac{-3Y+X}{6(X-Y)^2}$	$\frac{-3Y+X}{6(X-Y)^2}$	0
$I_3^n(1,3,3;0,X,Y)$	$-\frac{1}{6(X-Y)}$	$-\frac{1}{6(X-Y)}$	0
$I_3^n(2,3,3;0,X,Y)$	$-\frac{1}{6(Y-X)}$	$-\frac{1}{6(Y-X)}$	0
$I_3^n(1,2,3;0,X,Y)$	$\frac{(X+Y)}{6(X-Y)^2}$	$\frac{(X+Y)}{6(X-Y)^2}$	0

Table 2: Rational and pole parts of three-point functions with one light-like leg and up to three Feynman parameters in the numerator. $\mathcal{U}$ is the operator extracting the pole and rational parts of the integral. $\mathcal{V}$ denotes the value obtained from the corresponding expression for the off-shell integral, in the limit where one leg goes on-shell. W is the difference $\mathcal{U}[I]-\mathcal{V}$ where the pole terms, absorbed into $\mathcal{U}[I_2^n]$, have been set to zero.

For three-point functions with one non-zero invariant X, we labelled the internal propagators in such a way that $\mathcal{S}_{13}=X$ and $\mathcal{S}_{12}=\mathcal{S}_{23}=0$, where $\mathcal{S}$ is defined in eq. (2.4). For three-point functions with two non-zero invariants X and Y, we set $\mathcal{S}_{23}=X$ and $\mathcal{S}_{13}=Y$. Thus the integrals $I_3^n(i,j,\ldots;0,0,X)$ are symmetric under exchange of $1\leftrightarrow 3$ and the

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	$\mathcal{U}[I]$	$\mathcal{V}$	W
$I_3^{n+2}(0,0,X)$	$-\frac{1}{2}\mathcal{U}[I_2^n]-\frac{1}{2}$	$-\frac{1}{2}\mathcal{U}[I_2^n]-\frac{1}{2}$	0
$I_3^{n+2}(1;0,0,X)$	$-\frac{1}{6}\mathcal{U}[I_2^n]-\frac{1}{9}$	$-\frac{1}{6}\mathcal{U}[I_2^n]-\frac{1}{9}$	0
$I_3^{n+2}(2;0,0,X)$	$-\frac{1}{6}\mathcal{U}[I_2^n]-\frac{5}{18}$	$-\frac{1}{6}\mathcal{U}[I_2^n]-\frac{5}{18}$	0
$I_3^{n+2}(3;0,0,X)$	$-\frac{1}{6}\mathcal{U}[I_2^n]-\frac{1}{9}$	$-\frac{1}{6}\mathcal{U}[I_2^n]-\frac{1}{9}$	0
$I_3^{n+2}(0,X,Y)$	$-\frac{1}{2}\mathcal{U}[I_2^n]-\frac{1}{2}$	$-\frac{1}{2}\mathcal{U}[I_2^n]-\frac{1}{2}$	0
$I_3^{n+2}(1;0,X,Y)$	$-\frac{1}{6}\mathcal{U}[I_2^n]+\frac{-5X+2Y}{18(X-Y)}$	$-\frac{1}{6}\mathcal{U}[I_2^n]+\frac{-5X+2Y}{18(X-Y)}$	0
$I_3^{n+2}(2;0,X,Y)$	$-\frac{1}{6}\mathcal{U}[I_2^n]+\frac{-5Y+2X}{18(Y-X)}$	$-\frac{1}{6}\mathcal{U}[I_2^n]+\frac{-5X+2Y}{18(X-Y)}$	0
$I_3^{n+2}(3;0,X,Y)$	$-\frac{1}{6}\mathcal{U}[I_2^n]-\frac{1}{9}$	$-\frac{1}{6}\mathcal{U}[I_2^n]-\frac{1}{9}$	0

Table 3: Rational and pole parts of $(n+2)$-dimensional three-point functions with one or two light-like legs and up to one Feynman parameter in the numerator. $\mathcal{U}$ is the operator extracting the pole and rational parts of the integral. Note that the poles in I_3^{n+2} are of ultraviolet nature, but we do not distinguish the nature of the poles denoted by $\mathcal{P}[I_2^n]$. $\mathcal{V}$ denotes the value obtained from the corresponding expression for the off-shell integral, in the limit where one or two legs go on-shell. W is the difference $\mathcal{U}[I]-\mathcal{V}$, where the pole terms, absorbed into $\mathcal{U}[I_2^n]$, are set to zero.

$I_3^n(i,j,\ldots;0,X,Y)$ are symmetric under simultaneous exchange of $1\leftrightarrow 2$ and $X\leftrightarrow Y$.

In the tables, the results for the divergent three-point functions with Feynman parameters in the numerator are split up in the following way:

$$\mathcal{U}[I_3^n(\{j_l\})] = d_3\,\mathcal{U}[I_3^n(0,0,X)] + d_2\,\mathcal{U}[I_2^n] + \mathcal{V}(\{j_l\}) + W(\{j_l\})\,, \tag{A.6}$$

where d_3 is equal to one if the integral has a double pole, and zero otherwise. The functions $\mathcal{V}(\{j_l\})$ and $W(\{j_l\})$ denote the remaining finite part, where $\mathcal{V}(\{j_l\})$ is the part which is equal to the limit of the corresponding expression for the off-shell integral (given in eqs. (2.26),(2.27)) when one or two invariants go to zero, and $W(\{j_l\})$ is the finite remainder, i.e. the *difference* to the on-shell limit of the corresponding off-shell integral, after having singled out the pole contributions. To give an example, we have

$$\begin{aligned}
\mathcal{U}[I_3^n(2,2,2;0,0,s_1)] &= \mathcal{U}[I_3^n(0,0,s_1)] + \frac{11}{3s_1}\,\mathcal{U}[I_2^n] + \frac{19}{9s_1} \\
\lim_{s_2,s_3\to 0}\mathcal{U}[I_3^n(2,2,2;s_3,s_2,s_1)] &= -\lim_{s_2,s_3\to 0}\mathcal{U}[A_{222}^{3,3}] = \frac{2}{s_1} = \mathcal{V}(2,2,2;s_1) \\
&\Rightarrow W(2,2,2;s_1) = \frac{19}{9s_1} - \mathcal{V}(2,2,2;s_1) = \frac{1}{9s_1}\,.
\end{aligned} \tag{A.7}$$

The fact that W is not always zero of course does not mean that the results obtained by procedures (A.1) and (A.2) have to be different. The coefficients of the corresponding integrals as well as the number of non-zero two-point functions are different in the two

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approaches, such that after summation over all contributions making up a finite amplitude, the results will be the same. This has been checked explicitly by calculating the 4-photon and the 6-photon amplitudes in both ways.

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